Lecture 10: Increasing linearly independent sets, basis, and dimension

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Theorem:
Let $\{v_1, \ldots, v_m\}$ LI and consider span$\{v_1, \ldots, v_m\}$
Then,
$\{v, v_1, \ldots, v_m\}$ LI <=> $v \notin \text{span}\{v_1, \ldots, v_m\}$

PROOF
"=>" By observation 8 (last lecture), none of v, v ,...,v is contained in the span of the remaining ones. In particular, v is not.
"<=" Consider $av + a_1 v_1 + \cdots + a_m v_m = 0$. If $a \neq 0$, then $v = -\frac{a_1}{a} v_1 - \cdots - \frac{a_m}{a} v_m$
This is not possible, so $a = 0$. But, since $\{v_1, \ldots, v_m\}$ is linearly independent,
$a_1 = \cdots = a_m = 0$.

<u>NOTE</u>: We can increase any linearly independent set as long as it does not already span the vector space.

<u>Examples</u>:

a) $\mathbb{P}_2 : \{x^2, 1 + 2x\}$ LI
because:
should be zero everywhere
$ax^2 + b(2x + 1) = ax^2 + 2bx + b =: p(x) \equiv 0$
If $a \neq 0$, then p(x) could have at most two roots. So, $a = 0$. Moreover $p(0) = b$. So $b = 0$.

$x^3 \notin \text{span}\{x^2, 1 + 2x\}$
Hence, $\{x^3, x^2, 1 + 2x\}$ is linearly independent.

b) $M_{22}(\mathbb{R}) : \left\{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\right\}$ LI
<u>because</u>:
$a\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + b\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} = \begin{bmatrix}a & b\\0 & 0\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$
This only works with $a = b = 0$.
$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \notin \text{span}\left\{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\right\}$
Hence, $\left\{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right\}$ is linearly independent.

c) $\mathbb{R}^4$: $\{(1,2,1,1), (1,3,5,6)\}$ LI
$(1,0,0,0) \notin \text{span}\{(1,2,1,1), (1,3,5,6)\}$
<u>because</u>
$(1,0,0,0) = a(1,2,1,1) + b(1,3,5,6)$

$1 = a + b$
$0 = 2a + 3b$
$0 = a + 5b$
$0 = a + 6b$
contradiction

Hence, $\{(1,2,1,1), (1,3,5,6), (1,0,0,0)\}$ is linearly independent.

The most difficult part of this process is finding a vector that is not an element of span{...}

<u>9 Basis and Dimension</u>

Chapter 8 Recap

- Any linearly independent spanning set can be reduced by removing a vector that's in the span of the rest.
- Any linearly independent set in a space W can be increased by adding a vector that isn't in the span of the rest of the vectors (in other words: by adding a vector that **is not** a linear combination of any of the other vectors in the set)

Theorem relating linearly independent sets to spanning sets

If S is a spanning set, any larger set (containing S) must be linearly dependent.

- ie. if a vector space V can be spanned by n vectors, any LI subset has at most n vectors
- ie. if V has a subset of m linearly independent vectors, then any spanning set has at least m vectors
- ie. the size of any linearly independent set in $V \leq$ the size of any spanning set of V
  - this theorem applies to any vector space, including subspaces

A **basis** of V is defined as any linearly independent spanning set in V, while the **dimension** of V, denoted $\dim(V)$, is defined as the number of elements in any linearly independent spanning set (any basis) of V.

Examples:

a) $\mathbb{R}^3 = \text{span}\{(1,0,0), (0,1,0), (0,0,1)\}$
   a. any LI set in $\mathbb{R}^3$ has at most 3 vectors, so any set in $\mathbb{R}^3$ with at least 4 vectors is linearly dependent.

b) $M_{22}(\mathbb{R}) = \text{span}\left\{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right\}$
   a. any LI set in $M_{22}(\mathbb{R})$ has at most 4 vectors. Any set in $M_{22}(\mathbb{R})$ with at least 5 vectors is linearly dependent.

c) $U = \{(x, y, 0) \mid x, y \in \mathbb{R}\} = \text{span}\{(1,0,0), (0,1,0)\} \subseteq \mathbb{R}^3$
   a. any LI set in U has at most 2 vectors, even though there are linearly independent sets with 3 vectors in the surrounding space $\mathbb{R}^3$